Step III, Hints and Answers
June 2005

STEP Mathematics III 2005: Hints and Answers

Section A: Pure Mathematics

1 To prove the first part, use the results: $\cos B = \sin\left(\frac{\pi}{2} - B\right)$, whatever the value of B; and

$$\sin A = \sin Y \Leftrightarrow A = Y + 2n\pi \text{ or } A = \pi - Y + 2n\pi;$$

thus here, replacing Y by $\frac{\pi}{2} - B$, $A = 2n\pi + \frac{\pi}{2} \pm B$.

For the next part, it is probably easiest to use the fact that $a\sin x \pm b\cos x$ can be written in the form $R\sin(x \pm \alpha)$; here,

$$\sin x \pm \cos x = \sqrt{2}\left(\sin x \cos\frac{\pi}{4} \pm \cos x \sin\frac{\pi}{4}\right) = \sqrt{2}\sin\left(x \pm \frac{\pi}{4}\right)$$

so $|\sin x \pm \cos x| \leqslant \sqrt{2}$.

Now, from the first part,

$$\sin(\sin x) = \cos(\cos x) \Leftrightarrow \sin x = 2n\pi + \frac{\pi}{2} \pm \cos x$$

so

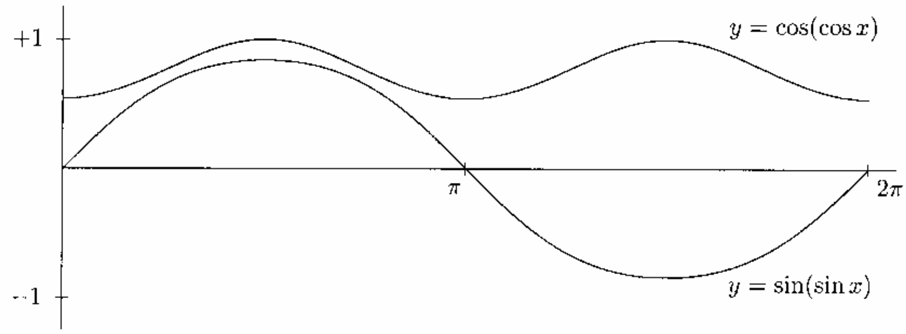
$$|\sin x \pm \cos x| \geqslant \left|2n\pi + \frac{\pi}{2}\right| \geqslant \frac{\pi}{2} > \sqrt{2},$$

which is a contradiction.

All the curves asked for have period 2π, so they will be sketched for x in this range only.

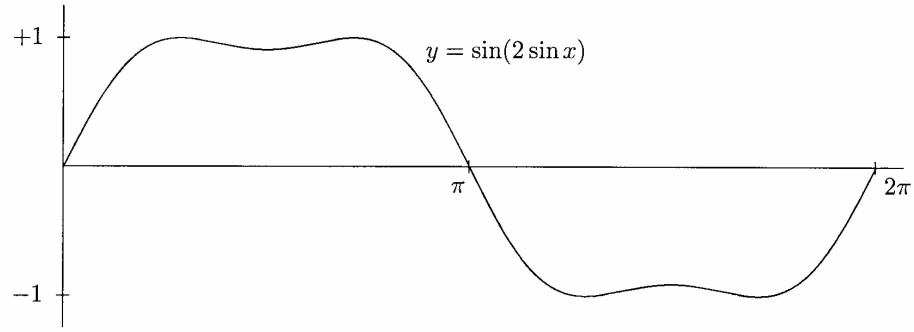
For $y = \sin(\sin x)$, $y = 0$ when $\sin x = 0$ only (since $|\sin x| < \pi$), so at 0, π and 2π; the turning points are at $\cos x \cos(\sin x) = 0$, so when $\cos x = 0$, that is at $x = \frac{\pi}{2}, \frac{3\pi}{2}$, or when $\cos(\sin x) = 0$,which is impossible since $|\sin x| < \frac{\pi}{2}$; the turning points are a maximum at $\left(\frac{\pi}{2}, \sin(1)\right)$ and a minimum at $\left(\frac{3\pi}{2}, -\sin(1)\right)$, where $\sin(1) \approx 0{\cdot}84$.

For $y = \cos(\cos x)$, $y > 0$ for all x, since $|\cos x| \leqslant 1 < \frac{\pi}{2}$; the turning points are at $\sin x \sin(\cos x) = 0$, so when either $\sin x = 0$ or $\cos x = 0$, that is at $x = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$; the turning points are maxima at $\left(\frac{\pi}{2}, 1\right)$ and $\left(\frac{3\pi}{2}, 1\right)$, and minima at $(0, \cos(1))$, $(\pi, \cos(1))$, $(2\pi, \cos(1))$, where $\cos(1) \approx 0{\cdot}54$.



STEP Mathematics III 2005: Hints and Answers

For the curve $y = \sin(2\sin x)$, $y = 0$ if $2\sin x$ is a multiple of π, which is only possible if $\sin x = 0$ (since $|2\sin x| < \pi$), so when x is 0, π and 2π; the turning points are at $2\cos x\cos(2\sin x) = 0$; so when $\cos x = 0$, that is at $x = \frac{\pi}{2}, \frac{3\pi}{2}$, or when $2\sin x$ is an odd multiple of $\frac{\pi}{2}$, which is only possible if $2\sin x = \pm\frac{\pi}{2}$, so when $\sin x = \frac{\pi}{4} \approx \pm 0{\cdot}8$; the turning points are a minimum at $\left(\frac{\pi}{2}, \sin 2\right)$, where $\sin 2 \approx 0{\cdot}91$ and maxima either side of this, with y-coordinates 1 and a maximum at $\left(\frac{3\pi}{2}, -\sin 2\right)$ with minima either side with y-coordinates -1.



2 This equation can be solved by separating the variables:

$$\int \frac{2\,\mathrm{d}y}{y} = -\int \frac{2x\,\mathrm{d}x}{x^2+a^2} \quad \text{so} \quad \ln(y^2) = -\ln(x^2+a^2) + k \quad \text{or} \quad y^2(x^2+a^2) = c^2.$$

The curve has two branches: one has $y > 0$, reflection symmetry about the y-axis, a maximum at $\left(0, \frac{c}{a}\right)$ and $y \longrightarrow 0$ as $|x| \longrightarrow \infty$; the other has $y < 0$ and is a reflection of the first branch in the x-axis.

$$\frac{\mathrm{d}}{\mathrm{d}x}(x^2+y^2) = 2x - 2y\frac{\mathrm{d}y}{\mathrm{d}x} = 2x - \frac{2xy^2}{x^2+a^2} = 2x - \frac{2xc^2}{(x^2+a^2)^2}$$

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}(x^2+y^2) = 2 - \frac{2c^2}{(x^2+a^2)^2} + \frac{4xc^2\cdot 2x}{(x^2+a^2)^3} = 2\left(1 - \frac{c^2}{(x^2+a^2)^2}\right) + \frac{8c^2x^2}{(x^2+a^2)^3}$$

STEP Mathematics III 2005: Hints and Answers

(i) $\dfrac{d}{dx}(x^2+y^2)=0$ when $x=0$ and when $c^2=(x^2+a^2)^2$, but the latter is not possible if $0<c<a^2$. If $x=0$, $y=\pm\dfrac{c}{a}$ and $\dfrac{d^2}{dx^2}(x^2+y^2)=1-\dfrac{c^2}{a^4}$ which is positive if $0<c<a^2$, indicating a local minimum. Hence the points on the curve whose distance from the origin is least are $\left(0\,,\,\pm\dfrac{c}{a}\right)$.

(ii) If $c>a^2$ then $\dfrac{d^2}{dx^2}(x^2+y^2)$ is negative at $x=0$, indicating a local maximum there; but in this case there are further stationary points at $x^2=c-a^2\,,\ y=\pm\sqrt{c}$ and at these points $\dfrac{d^2}{dx^2}(x^2+y^2)=\dfrac{8x^2}{c}>0$. Hence the points on the curve whose distance from the origin is least are $(\pm\sqrt{c-a^2}\,,\ \pm\sqrt{c})$.

3 Direct substitution gives

$$f(g(x))=(x^2+rx+s)^2+p(x^2+rx+s)+q$$

$$=x^4+2rx^3+(r^2+2s+p)x^2+(2rs+rp)x+s^2+ps+q.$$

If $x^4+ax^3+bx^2+cx+d$ is to have this form then it is necessary to choose $r=\dfrac{a}{2}$ and to choose s and p to satisfy $2s+p=b-r^2=b-\dfrac{a^2}{4}$ and $r(2s+p)=c$ or $a(2s+p)=2c$. Thus $a\left(b-\dfrac{a^2}{4}\right)=2c$ is a necessary condition for this to be possible.

It is also sufficient: in fact, pick $p=0$; then $s=\dfrac{4b-a^2}{8}$ and $q=d-s^2$ will do.

Expanding the second form gives

$$(x^2+vx+w)^2-k=x^4+2vx^3+(v^2+2w)x^2+2vwx+w^2-k,$$

but this is identical to $x^4+2rx^3+(r^2+2s+p)x^2+(2rs+rp)x+s^2+ps+q$ with $p=0$, $v=r$, $w=s$ and $k=-q$, and so, since the sufficiency demonstrated above allowed the choice $p=0$, the condition is the same.

To solve the final equation, write the quartic in the second form:

$$x^4-4x^3+10x^2-12x+4=(x^2-2x+3)^2-5=0$$

so

$$x^2-2x+3-\sqrt{5}=0 \text{ or } x^2-2x+3+\sqrt{5}=0$$

so

$$x=1\pm\sqrt{\sqrt{5}-2} \text{ or } 1\pm j\sqrt{\sqrt{5}+2}.$$

STEP Mathematics III 2005: Hints and Answers

4 For the base case you need to verify that $u_{2n} = \frac{b}{a}u_{2n-1}$ and $u_{2n+1} = cu_{2n}$ when $n = 1$:

$u_1 = a$, $u_2 = b$ so $u_{2n} = \frac{b}{a}u_{2n-1}$ when $n = 1$;

$u_3 = \frac{u_2}{u_1}(ku_1 - u_2) = u_2\frac{ka-b}{a}$ so $u_{2n+1} = cu_{2n}$ when $n = 1$, provided $c = k - \frac{b}{a}$.

For the induction step, assume that $u_{2n} = \frac{b}{a}u_{2n-1}$ and $u_{2n+1} = cu_{2n}$ when $n = N$ then

$$\begin{aligned} u_{2N+2} &= \frac{u_{2N+1}}{u_{2N}}(ku_{2N} - u_{2N+1}) \\ &= u_{2N+1}(k-c) \text{ (by the induction hypothesis)} \\ &= \frac{b}{a}u_{2N+1} \text{ (by the definition of } c) \end{aligned}$$

and

$$\begin{aligned} u_{2N+3} &= \frac{u_{2N+2}}{u_{2N+1}}(ku_{2N+1} - u_{2N+2}) \\ &= u_{2N+2}\left(k - \frac{b}{a}\right) \text{ (by what has just been shown)} \\ &= cu_{2N+2}. \end{aligned}$$

which completes the induction.

Hence $u_{2n} = \frac{bc}{a}u_{2n-2} = \ldots = \left(\frac{bc}{a}\right)^{n-1} u_2 = b\left(\frac{bc}{a}\right)^{n-1}$

and $u_{2n-1} = \frac{bc}{a}u_{2n-3} = \ldots = \left(\frac{bc}{a}\right)^{n-1} u_1 = a\left(\frac{bc}{a}\right)^{n-1}$

(i) For u_n to be geometric requires $\frac{u_{2n}}{u_{2n-1}} = \frac{u_{2n+1}}{u_{2n}}$; that is, $\frac{b}{a} = c = k - \frac{b}{a}$ or $ak = 2b$;

(ii) For u_n to have period 2 requires $u_{2n+1} = u_{2n-1}$, but $u_{2n+1} = cu_{2n} = \frac{cb}{a}u_{2n-1}$, so it is necessary that $\frac{cb}{a} = 1$ or $a^2 + b^2 = kab$;

(iii) For u_n to have period 4 requires $u_{2n+3} = u_{2n-1}$ so, by the previous part, it is necessary that $\left(\frac{cb}{a}\right)^2 = 1$ but $\frac{cb}{a} \neq 1$ (to avoid period 2) so $\frac{cb}{a} = -1$ or $b^2 - a^2 = kab$.

STEP Mathematics III 2005: Hints and Answers

5 The point on the curve with the required gradient is given by

$$\frac{dy}{dx} = 2ax + b = m \text{ or } x = \frac{m-b}{2a},$$

with

$$y = a\left(\frac{m-b}{2a}\right)^2 + b\left(\frac{m-b}{2a}\right) + c = \frac{m^2-b^2}{4a} + c.$$

The equation of the tangent is therefore:

$$y - mx = a\left(\frac{m-b}{2a}\right)^2 + b\left(\frac{m-b}{2a}\right) + c - m\left(\frac{m-b}{2a}\right)$$

$$= c - \frac{(m-b)}{2a}\left(m - b - \frac{(m-b)}{2}\right) = c - \frac{(m-b)^2}{4a}.$$

The curves have a common tangent with gradient m if and only if the equations of the tangents to the two curves with gradient m are identical; that is, have the same intercept, so if and only if

$$c_1 - \frac{(m-b_1)^2}{4a_1} = c_2 - \frac{(m-b_2)^2}{4a_2}$$

that is,

$$4a_1a_2c_1 - a_2m^2 + 2a_2b_1m - a_2b_1^2 = 4a_1a_2c_2 - a_1m^2 + 2a_1b_2m - a_1b_2^2$$

which gives the quoted result.

There is exactly one common tangent when $a_1 \neq a_2$ when the quadratic equation for m has exactly one root, which occurs if and only if the discriminant of the equation is zero; that is

$$4(a_1b_2 - a_2b_1)^2 = 4(a_2 - a_1)(4a_1a_2(c_2 - c_1) + a_2b_1^2 - a_1b_2^2)$$

$$\Leftrightarrow \quad 4a_1^2b_2^2 - 8a_1a_2b_1b_2 + 4a_2^2b_1^2 = 16a_1a_2(a_2-a_1)(c_2-c_1) + 4a_2^2b_1^2 + 4a_1^2b_2^2 - 4a_1a_2(b_1^2 + b_2^2)$$

$$\Leftrightarrow \quad 4a_1a_2(b_1^2 + b_2^2) - 8a_1a_2b_1b_2 = 16a_1a_2(a_2 - a_1)(c_2 - c_1)$$

$$\Leftrightarrow \quad b_1^2 + b_2^2 - 2b_1b_2 = 4(a_2 - a_1)(c_2 - c_1) \quad \text{(dividing by } 4a_1a_2 \text{ which is non-zero).}$$

The curves touch if there is exactly one solution to the simultaneous equations

$$y = a_1x^2 + b_1x + c_1 \text{ and } y = a_2x^2 + b_2x + c_2;$$

that is, if the equation $(a_2 - a_1)x^2 + (b_2 - b_1)x + (c_2 - c_1) = 0$ has exactly one root so, again using the discriminant condition, if and only if $(b_2 - b_1)^2 = 4(a_2 - a_1)(c_2 - c_1)$, which is the same condition.

If $a_1 = a_2$ the curves have exactly one common tangent if there is exactly one solution to

$$2m(b_2 - b_1) + 4a(c_2 - c_1) + (b_1^2 - b_2^2) = 0;$$

since this is just a linear equation, the only condition is that $b_2 - b_1 \neq 0$.

STEP Mathematics III 2005: Hints and Answers

6 Direct substitution of $x = 2a\cosh\left(\frac{T}{3}\right)$ into the left hand side of the equation gives

$$\left(2a\cosh\left(\frac{T}{3}\right)\right)^3 - 6a^3\cosh\left(\frac{T}{3}\right) = 2a^3\left(4\left(\cosh\left(\frac{T}{3}\right)\right)^3 - 3\cosh\left(\frac{T}{3}\right)\right) = 2a^3\cosh T$$

(by the first result given at the start of the question).

Let $a^2 = b$, which is possible since $b > 0$, and $\cosh T = \frac{c}{a^3}$, which requires $\frac{c}{a^3} \geqslant 1$; but this holds if you choose a to have the same sign as c, since then $\frac{c}{a^3} > 0$ and $c^2 > b^3 = a^6$.

Then, by the second result given at the start of the question,

$$T = \ln\left(\frac{c}{a^3} + \sqrt{\frac{c^2}{a^6} - 1}\right) = \ln\left(\frac{c + \sqrt{c^2 - b^3}}{a^3}\right) = 3\ln\left(\frac{u}{a}\right),$$

so one of the roots of the equation $x^3 - 3bx = 2c$ is

$$2a\cosh\left(\ln\left(\frac{u}{a}\right)\right) = 2a\,\frac{\frac{u}{a} + \frac{a}{u}}{2} = u + \frac{b}{u}.$$

Note that, since $u + \frac{b}{u}$ is a root of the equation $x^3 - 3bx = 2c$,

$$2c = \left(u + \frac{b}{u}\right)^3 - 3b\left(u + \frac{b}{u}\right) = \left(u + \frac{b}{u}\right)\left(u^2 + \frac{b^2}{u^2} - b\right)$$

and that

$$u^2 + \frac{b^2}{u^2} - b - \left(u + \frac{b}{u}\right)^2 = -3b,$$

so

$$x^3 - 3bx - 2c = \left(x - \left(u + \frac{b}{u}\right)\right)\left(x^2 + \left(u + \frac{b}{u}\right)x + u^2 + \frac{b^2}{u^2} - b\right),$$

so the other two roots of $x^3 - 3bx = 2c$ are the roots of $x^2 + \left(u + \frac{b}{u}\right)x + u^2 + \frac{b^2}{u^2} - b = 0$, which are

$$\tfrac{1}{2}\left(-\left(u + \frac{b}{u}\right) \pm \sqrt{\left(u + \frac{b}{u}\right)^2 - 4\left(u^2 + \frac{b^2}{u^2} - b\right)}\right)$$

$$= \tfrac{1}{2}\left(-\left(u + \frac{b}{u}\right) \pm \sqrt{-3\left(u^2 + \frac{b^2}{u^2} - 2b\right)}\right) = \tfrac{1}{2}\left(-\left(u + \frac{b}{u}\right) \pm \sqrt{3}j\left(u - \frac{b}{u}\right)\right),$$

that is $\omega u + \omega^2\frac{b}{u}$ and $\omega^2 u + \omega\frac{b}{u}$.

In $x^3 - 6x = 6$, $b = 2$, $c = 3$, so $a = \sqrt{2}$ and so $u = \sqrt[3]{3+1} = 2^{\frac{2}{3}}$ and $\frac{b}{u} = 2^{\frac{1}{3}}$, so the solutions are $2^{\frac{1}{3}} + 2^{\frac{2}{3}}$, $\omega 2^{\frac{1}{3}} + \omega^2 2^{\frac{2}{3}}$ and $\omega^2 2^{\frac{1}{3}} + \omega 2^{\frac{2}{3}}$.

STEP Mathematics III 2005: Hints and Answers

7 Substituting $u = x^m$ gives

$$\int \frac{m\mathrm{d}x}{x\,\mathrm{f}(x^m)} = \int \frac{mx^{m-1}\mathrm{d}x}{x^m\,\mathrm{f}(x^m)} = \int \frac{\mathrm{d}u}{u\,\mathrm{f}(u)} = \mathrm{F}(u) + c = \mathrm{F}(x^m) + c$$

(i) $$\int \frac{\mathrm{d}x}{x^n - x} = \int \frac{\mathrm{d}x}{x\,(x^{n-1} - 1)},$$

so letting $u = x^{n-1}$ and $\mathrm{f}(u) = u - 1$,

$$\int \frac{(n-1)\mathrm{d}x}{x^n - x} = \int \frac{\mathrm{d}u}{u\,(u-1)} = \int \frac{1}{u-1} - \frac{1}{u}\,\mathrm{d}u = \ln\left|\frac{u-1}{u}\right|$$

so $$\int \frac{\mathrm{d}x}{x^n - x} = \frac{1}{n-1}\ln\left|\frac{x^{n-1}-1}{x^{n-1}}\right| + c.$$

(ii) $$\int \frac{\mathrm{d}x}{\sqrt{x^n + x^2}} = \int \frac{\mathrm{d}x}{x\sqrt{x^{n-2}+1}} \quad \text{(for } x > 0\text{)}$$

so letting $u = x^{n-2}$ and $\mathrm{f}(u) = \sqrt{u+1}$ (and assuming $n \neq 2$)

$$\int \frac{(n-2)\mathrm{d}x}{\sqrt{x^n + x^2}} = \int \frac{\mathrm{d}u}{u\sqrt{u+1}}.$$

Substituting $u = v^2 - 1$ with $v > 0$,

$$\int \frac{\mathrm{d}u}{u\sqrt{u+1}} = \int \frac{2v\mathrm{d}v}{(v^2-1)\,v} = \int \frac{1}{v-1} - \frac{1}{v+1}\,\mathrm{d}v = \ln\left|\frac{v-1}{v+1}\right|$$

so $$\int \frac{\mathrm{d}x}{\sqrt{x^n + x^2}} = \frac{1}{n-2}\ln\left|\frac{\sqrt{x^{n-2}+1}-1}{\sqrt{x^{n-2}+1}+1}\right| + c.$$

STEP Mathematics III 2005: Hints and Answers

8 Direct use of the important result $|z|^2 = zz^*$ gives

$$|a-c|^2 = (a-c)(a^* - c^*) = aa^* + cc^* - ac^* - ca^*.$$

OAC is a right angle if and only if $|\mathrm{AC}|^2 + |\mathrm{OA}|^2 = |\mathrm{OC}|^2$; that is, $|a-c|^2 + |a|^2 = |c|^2$ or, using the result above, $2aa^* - ac^* - ca^* = 0$.

The circle has centre C and radius AC, so complex numbers representing points on the circle satisfy $|z-c|^2 = |a-c|^2$ or $zz^* - zc^* - cz^* = aa^* - ac^* - ca^*$.
Because OA is a tangent to the circle, angle OAC is a right angle and so $2aa^* - ac^* - ca^* = 0$ as above; thus the condition for points to lie on the circle becomes $zz^* - zc^* - cz^* + aa^* = 0$.

P lies on this circle if and only if

$$aba^*b^* - abc^* - ca^*b^* + aa^* = 0$$

and P$'$ lies on the circle if and only if

$$\frac{aa^*}{bb^*} - \frac{a}{b^*}c^* - c\frac{a^*}{b} + aa^* = 0$$

but multiplying this by bb^* (which is not equal to zero) gives the same condition.

Conversely, if the points lie on the circle represented by $|z-c|^2 = |a-c|^2$,

$$aba^*b^* - abc^* - ca^*b^* + aa^* = 2aa^* - ac^* - ca^*$$

and

$$\frac{aa^*}{bb^*} - \frac{a}{b^*}c^* - c\frac{a^*}{b} = 2aa^* - ac^* - ca^*,$$

so that

$$aba^*b^* - abc^* - ca^*b^* + aa^* = bb^*(2aa^* - ac^* - ca^*)$$

and so, provided $bb^* \neq 1$, it must be the case that $2aa^* - ac^* - ca^* = 0$, and this shows that OAC is a right angle and hence that OA is a tangent to the circle.

STEP Mathematics III 2005: Hints and Answers

Section B: Mechanics

9 Let the speeds of A and B after the first collision be u_1, u_2, then conservation of momentum gives

$$4eu_1 + (1-e)^2u_2 = 4e(1-e)v - (1-e)^2 \cdot 2ev = 2ev(1-e^2)$$

and the restitution equation gives

$$u_2 - u_1 = (1-e)v + 2ev = e(1+e)v.$$

To find u_1, multiply the second equation by $(1-e)^2$ and subtract it from the first:

$$u_1 = \frac{2ev(1-e^2) - (1-e)^2e(1+e)v}{4e + (1-e)^2} = \frac{ev(1-e^2)}{(1+e)^2}\,(2-(1-e)) = e(1-e)v$$

To find u_2, multiply the second equation by $4e$ and add it to the first:

$$u_2 = \frac{2ev(1-e^2) + 4e^2(1+e)v}{4e + (1-e)^2} = \frac{2ev(1+e)}{(1+e)^2}\,((1-e)+2e) = 2ev.$$

After B strikes the vertical wall, it rebounds with speed $2e^2v$, so if x is the distance from the wall at which the second collision occurs, the total time between collisions is

$$\frac{d-x}{e(1-e)v} = \frac{d}{2ev} + \frac{x}{2e^2v},$$

so that $2e(d-x) = (1-e)(ed+x)$ or $x = ed$.
Note that the situation is now that given initially, with all distances and speeds scaled by e. Thus the n^{th} collision occurs a distance de^{n-1} from the wall, and the speed of A between the n^{th} and the $(n+1)^{\text{th}}$ collisions is $(1-e)ve^n$, so the time between collisions is

$$\frac{de^{n-1} - de^n}{(1-e)ve^n} = \frac{d}{ev},$$

which is independent of n.

STEP Mathematics III 2005: Hints and Answers

10 When the discs are a distance $2x$ apart, their centres are $2(x+r)$ apart and the length of the band is $4(x+r)+2\pi r$. Therefore the tension in the band is

$$T(x) = 2\left(\frac{\pi mg}{12}\right)\frac{4x+4r}{2\pi r} = \frac{mg}{6r}(x+r),$$

and hence the force on each disc is $F(x) = 2T(x) = \dfrac{mg}{3r}(x+r)$; the elastic potential energy stored in the band is

$$E(x) = \tfrac{1}{2}\left(\frac{\pi mg}{12}\right)\frac{(4x+4r)^2}{2\pi r} = \frac{mg}{3r}(x+r)^2.$$

(i) The maximum frictional resistance to the motion of a disc is μmg, so for the disc to start sliding requires $F(2r) > \mu mg$, that is $1 > \mu$. For the disc then to come to rest before a collision occurs, the elastic energy released by the band as x decreases from $2r$ to 0 must be insufficient to do the work against friction required to bring the discs into contact. This required work is $2r\mu mg$ for each disc, so $4r\mu mg$ in total, so the condition you need is $E(2r) - E(0) < 4r\mu mg$; that is $4r\mu > 3r - \dfrac{1}{3}r$ or $\mu > \frac{2}{3}$.

(ii) By the argument in (i), $E(2r) = E(0) + K + 4r\mu mg$, where K is the kinetic energy just before collision, so

$$K = 3rmg - \frac{1}{3}rmg - 4\mu rmg = mgr\left(\frac{8}{3} - 4\mu\right).$$

(iii) Notice first that for the discs to come to rest after the first collision, it is necessary that the discs collide, so $\mu^2 < \frac{4}{9}$, by part (i).

In order that the discs do not begin to move again, once they have come to rest for the first time after collision, each must come to rest at a point where $F(x) < \mu mg$, that is $x < (3\mu - 1)r$.

The value of x at which the particles do come to rest is given by the requirement that the loss in elastic and kinetic energy from the point of collision to the point where the discs are $2x$ apart is equal to the work done against friction on both particles in moving from 0 to $2x$ separation, that is $E(0) + \frac{1}{2}K - E(x) = 2x\mu mg$ or

$$\begin{aligned} 0 &= \frac{mgr}{3} + mgr\left(\frac{4}{3} - 2\mu\right) - \frac{mg}{3r}(x+r)^2 - 2x\mu \\ &> \frac{mgr}{3} + mgr\left(\frac{4}{3} - 2\mu\right) - \frac{mg}{3r}(3\mu)^2 - 2mg\mu(3\mu - 1)r \end{aligned}$$

using the inequality on x, so

$$0 > \frac{mgr}{3}\left(5 - 27\mu^2\right)$$

or $\mu^2 > \frac{5}{27}$.

STEP Mathematics III 2005: Hints and Answers

11 The energy of the system is the sum of the gravitational potential energy (GPE) and the kinetic energy (KE), with KE $= \frac{1}{2}m\left(a^2+b^2+c^2\right)\dot{\theta}^2$ and, taking the zero of GPE to be at the height of the spindle,

$$\begin{aligned}\text{GPE} &= mg\left(a\cos\theta - b\cos\left(\frac{\pi}{3}-\theta\right) - c\cos\left(\frac{\pi}{3}+\theta\right)\right)\\ &= \tfrac{1}{2}mg\left((2a-b-c)\cos\theta - (b-c)\sqrt{3}\sin\theta\right)\end{aligned}$$

simplifying using the $\cos(A \pm B)$ identities and the exact values of $\sin\frac{\pi}{3}$ and $\cos\frac{\pi}{3}$.

Equilibrium occurs at a stationary point of the GPE:

$$\frac{\text{dGPE}}{\text{d}\theta} = \tfrac{1}{2}mg\left(-(2a-b-c)\sin\theta - (b-c)\sqrt{3}\cos\theta\right) = 0$$

or when the total moment about the spindle of the gravitational forces on the particles is zero:

$$mg\left(a\sin\theta + b\sin\left(\frac{\pi}{3}-\theta\right) - c\sin\left(\frac{\pi}{3}+\theta\right)\right) = 0$$

which simplifies to the same equation using the $\sin(A \pm B)$ identities.

This equation is satisfied if

$$\frac{\sin\theta}{\cos\theta} = \tan\theta = -\frac{(b-c)\sqrt{3}}{2a-b-c} < 0$$

which has two solutions between 0 and 2π unless $a = b = c = 0$. It is useful to realise here that $\tan\theta = -\frac{p}{q} \Rightarrow \sin\theta = \mp\frac{p}{h}$ and $\cos\theta = \pm\frac{q}{h}$, where h is the positive number with $h^2 = p^2 + q^2$.

In this case, $\sin\theta == -\frac{(b-c)\sqrt{3}}{2R}$ and $\cos\theta = \frac{(2a-b-c)}{2R}$ give one equilibrium and $\sin\theta = \frac{(b-c)\sqrt{3}}{2R}$ and $\cos\theta = -\frac{(2a-b-c)}{2R}$ the other, where

$$R^2 = \frac{1}{4}\left((2a-b-c)^2 + 3(c-b)^2\right) = \frac{1}{2}\left((a-b)^2 + (b-c)^2 + (c-a)^2\right)$$

The equilibrium is stable at a minimum of the GPE and unstable at a maximum. Since

$$\frac{\text{d}^2\text{GPE}}{\text{d}\theta^2} = \tfrac{1}{2}mg\left(-(2a-b-c)\cos\theta + (b-c)\sqrt{3}\sin\theta\right).$$

the first equilibrium position given above is unstable and the second is stable.

For the system to make complete revolutions, you need the KE at the point with maximum GPE to be greater than zero: that is, the difference in GPE between the two equilibria (which is twice the maximum GPE) is less than the KE at the point with minimum GPE.
If ω = angular velocity at stable equilibrium, you therefore require

$$\tfrac{1}{2}m\left(a^2+b^2+c^2\right)\omega^2 > mg\left((2a-b-c)\frac{(2a-b-c)}{2R} + (c-b)\sqrt{3}\frac{(c-b)\sqrt{3}}{2R}\right)$$

That is, $\left(a^2+b^2+c^2\right)\omega^2 > 2g\left(\frac{4R^2}{2R}\right) = 4gR$.

STEP Mathematics III 2005: Hints and Answers

Section C: Probability and Statistics

12 If $X = aT + b(T_1 + T_2 + T_3 + T_4)$ then

$$\mathrm{E}[X] = at + b(t_1 + t_2 + t_3 + t_4) = (a+b)t,$$

since $t_1 + t_2 + t_3 + t_4 = t$, by definition. You require $\mathrm{E}[X] = t$ which gives $a + b = 1$.

$$\mathrm{Var}[X] = a^2\mathrm{Var}[T] + b^2\mathrm{Var}[T_1] + b^2\mathrm{Var}[T_2] + b^2\mathrm{Var}[T_3] + b^2\mathrm{Var}[T_4],$$

assuming the errors made by the five timers are independent,

$$= (a^2 + 4b^2)\sigma^2 = (a^2 + 4(1-a)^2)\sigma^2 = (5a^2 - 8a + 4)\sigma^2 = \left(5\left(a - \frac{4}{5}\right)^2 + \frac{4}{5}\right)\sigma^2$$

which has a minimum value of $\frac{4}{5}\sigma^2$ when $a = \frac{4}{5}$; that is, when $X = \frac{1}{5}(4T + (T_1 + T_2 + T_3 + T_4))$.

Rearranging the identity $\mathrm{Var}[Y] = \mathrm{E}[Y^2] - \mathrm{E}[Y]^2$ gives $\mathrm{E}[Y^2] = \mathrm{Var}[Y] + \mathrm{E}[Y]^2$,
so if $Y = cT + d(T_1 + T_2 + T_3 + T_4)$ then

$$\mathrm{E}[Y^2] = (c^2 + 4d^2)\sigma^2 + (ct + d(t_1 + t_2 + t_3 + t_4))^2 = (c^2 + 4d^2)\sigma^2 + (c+d)^2t^2,$$

which is equal to σ^2 regardless of the true lap times if $c + d = 0$ and $1 = c^2 + 4d^2 = 5c^2$,
so that $c = \dfrac{1}{\sqrt{5}}$ and $Y^2 = \frac{1}{5}(T - (T_1 + T_2 + T_3 + T_4))^2$.

The timers could reasonably expect the true time for the race to lie within k estimated standard errors of the estimated value where, for instance, $k = 2$ or 3; so between

$$\tfrac{1}{5}(4T + (T_1 + T_2 + T_3 + T_4)) + k\sqrt{\tfrac{4}{5} \times \tfrac{1}{5}(T - (T_1 + T_2 + T_3 + T_4))^2}$$

and

$$\tfrac{1}{5}(4T + (T_1 + T_2 + T_3 + T_4)) - k\sqrt{\tfrac{4}{5} \times \tfrac{1}{5}(T - (T_1 + T_2 + T_3 + T_4))^2};$$

that is, between $220.1 + \dfrac{k}{5}$ and $220.1 - \dfrac{k}{5}$. For $k = 2$, this is the interval $[219{\cdot}7\,,\ 220{\cdot}5]$.

STEP Mathematics III 2005: Hints and Answers

13 The probability that the player wins exactly £3 is equal to the probability that the next 3 scores which lie in the range 0 to w are non zero, and the fourth score which lies in the range 0 to w is zero as the occurrence of outcomes which lead to the game continuing does not affect the final result.

Hence the probability that the player wins exactly £3 is equal to $\left(\dfrac{w}{w+1}\right)^3 \dfrac{1}{w+1}$.

Similarly, the probability that he wins exactly £r is $\left(\dfrac{w}{w+1}\right)^r \dfrac{1}{w+1}$ and so his expected winnings are

$$\sum_{r=0}^{\infty} r\left(\frac{w}{w+1}\right)^r \frac{1}{w+1} = \frac{w}{(w+1)^2}\sum_{r=1}^{\infty} r\left(\frac{w}{w+1}\right)^{r-1} = \frac{w}{(w+1)^2}\frac{1}{\left(1-\dfrac{w}{w+1}\right)^2} = w.$$

This calculation uses the result $\displaystyle\sum_{r=1}^{\infty} rp^{r-1} = \sum_{r=0}^{\infty} rp^{r-1} = \frac{1}{(1-p)^2}$, which you may know, or can be derived by noticing that $\displaystyle\sum_{r=0}^{\infty} rp^{r-1} = \frac{\mathrm{d}}{\mathrm{d}p}\left(\sum_{r=0}^{\infty} p^r\right)$ and that $\displaystyle\sum_{r=0}^{\infty} p^r = \frac{1}{1-p}$, recognising an infinite geometric series.

In a second game, consider the cards set out in the order in which they will be drawn. Then only $w+1$ cards are relevant, and the zero card is equally likely to be any of these, so that the probability that he wins exactly £r is $\dfrac{1}{w+1}$ (for $r = 1, 2, \dots w$) and so his expected winnings are now

$$\sum_{r=0}^{w} r\frac{1}{w+1} = \frac{1}{w+1}\sum_{r=0}^{w} r = \frac{1}{w+1}\tfrac{1}{2}w(w+1) = \tfrac{1}{2}w.$$

STEP Mathematics III 2005: Hints and Answers

14 The integral of the density function from 0 to infinity must equal 1:

$$1 = \int_0^\infty \frac{Ck^{a+1}x^a}{(x+k)^{2a+2}}\,\mathrm{d}x = Ck^{a+1}\frac{a!(2a-a)!}{(2a+1)!k^{2a-a+1}},$$

using the given result with $m = a$ and $n = 2a$

$$= C\frac{a!a!}{(2a+1)!} \text{ so } C = \frac{(2a+1)!}{a!a!}.$$

Use the substitution $x = \dfrac{k^2}{u}$:

$$\int_0^v \frac{x^a}{(x+k)^{2a+2}}\,\mathrm{d}x = \int_\infty^{\frac{k^2}{v}} \frac{k^{2a}}{u^a\left(\dfrac{k}{u}(u+k)\right)^{2a+2}}\,\frac{-k^2\mathrm{d}u}{u^2} = \int_{\frac{k^2}{v}}^\infty \frac{u^a}{(u+k)^{2a+2}}\,\mathrm{d}u$$

Choosing $v = k$ gives $\displaystyle\int_0^k \mathrm{f}(x)\,\mathrm{d}x = \int_k^\infty \mathrm{f}(x)\,\mathrm{d}x$, so k is the median, and

$$\mathrm{E}[V] = \int_0^\infty \frac{C\,k^{a+1}\,x^{a+1}}{(x+k)^{2a+2}}\,\mathrm{d}x = \frac{(2a+1)!}{a!\,a!}k^{a+1}\frac{(a+1)!\,(a-1)!}{(2a+1)!\,k^a} = k\left(\frac{a+1}{a}\right).$$

Notice that $T < t$ if and only if $V > \dfrac{s}{t}$, so that

$$\mathrm{P}(T<t) = \mathrm{P}\left(V > \frac{s}{t}\right) = \int_{\frac{s}{t}}^\infty \frac{C\,k^{a+1}\,x^a}{(x+k)^{2a+2}}\,\mathrm{d}x$$

and making the substitution $x = \dfrac{s}{u}$:

$$= \int_t^0 \frac{C\,k^{a+1}\,s^a}{u^a\left(\dfrac{k}{u}\left(\dfrac{s}{k}+u\right)\right)^{2a+2}}\,\frac{-s\,\mathrm{d}u}{u^2} = \int_0^t \frac{C\,u^a\left(\dfrac{s}{k}\right)^{a+1}}{\left(\dfrac{s}{k}+u\right)^{2a+2}}\,\mathrm{d}u$$

so the density function is $\dfrac{C\,u^a\left(\frac{s}{k}\right)^{a+1}}{\left(\frac{s}{k}+u\right)^{2a+2}}$, which is the same as that of V with k replaced by $\dfrac{s}{k}$.

This means that the median time is $\dfrac{s}{k}$ and that the expected time is $\dfrac{s}{k}\left(\dfrac{a+1}{a}\right)$ and hence median velocity $\times$ median time $= s$, but $\mathrm{E}[V]\,\mathrm{E}[T] = s\left(\dfrac{a+1}{a}\right)^2$, which is greater than s.